Frequency-Damping Resolution of the Unit Disc: A Wavelet Idea

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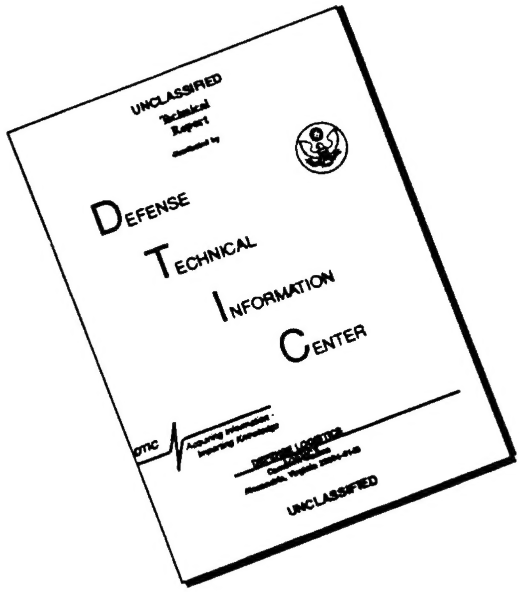

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Abstract

In this paper we introduce the numerical Laplace transform, a local time-frequency analysis method which applies to causal signals. The numerical Laplace transform resolves the identity, has good time-frequency resolution, and adapts resolution windows according to the time delay. The numerical Laplace transform is equivalent to a wavelet transform in the frequency domain. The discretized version of the numerical Laplace transform is invertible. The kernel vectors of the transform are frame vectors that are nearly tight over a fairly wide range of parameters. We demonstrate this with several numerical experiments. The numerical Laplace transform resolves a causal signal onto s-plane. With a suitable mapping, the signal is resolved into the frequency-damping unit disc.

1 Introduction

The literature of statistical signal processing has been dominated over the past twenty years by work on exact and subspace methods for identifying damped and undamped complex exponential modes [4, 3]. Such modes arise out of the physics of lumped linear systems, so they naturally carry physical information. Except for some recent extensions [6], the topic is well developed.

During the period of development of modal analysis, there has been renewed interest in methods of time-frequency analysis such as windowed Fourier analysis , Wigner distributions, wavelet decompositions, and other techniques for computing infinite-dimensional decompositions of signals. In a sense, these too are modal decompositions, for they decompose a signal into spectral modes that are localized in time and frequency or time and scale. However, it is usually impossible to make direct links between the time-frequency coefficients and physical phenomena. Windowed Fourier analysis does not help because the windowing and shifting process in the transform mechanism simply cut the signals into pieces. It is therefore desired to develop a time-frequency analysis method which bridges the gap between infinite-dimensional modal decompositions and finite-dimensional modal analysis.

2 Continuous Numerical Laplace Transform

We start from a wavelet transform in the frequency domain. Define the mother wavelet

$$\widehat{\psi_n}(j\nu) = \sqrt{\frac{2^n}{\Gamma(n)}}\frac{\Gamma(\frac{n+1}{2})}{(1+j\nu)^{\frac{n+1}{2}}} \quad n>1 \qquad (1)$$

with corresponding time domain window function $\psi_n(t)$:

$$\psi_n(t) = \gamma^{1/2}(t;n,2) \qquad (2)$$

$$\gamma(t;n,\alpha) = \frac{\alpha}{\Gamma(n)}(\alpha t)^{n-1}e^{-\alpha t}u(t)$$

$u(t)$ is the unit step function, $\quad n\in\mathbf{N}, \quad \alpha>0$

Note that $\psi_n(t)$ has unit norm. From this mother wavelet, construct the basis of scaled translates

$$\left\{\sigma^{-\frac{1}{2}}\widehat{\psi_n}\left(j(\frac{\nu-\omega}{\sigma})\right), \sigma>0, -\infty<\omega<\infty\right\}$$

Then define the complex frequency $s=\sigma+j\omega$ and write the scaled translates of the basis as

$$\begin{aligned} & \sigma^{-\frac{1}{2}}\widehat{\psi_n}\left(j(\frac{\nu-\omega}{\sigma})\right) & (3)\\ = \; & \sigma^{-\frac{1}{2}}\frac{\Gamma(\frac{n+1}{2})}{\sqrt{2\Gamma(n)}}\left(\frac{2\sigma}{s^*+j\nu}\right)^{\frac{n+1}{2}} & (4)\\ = \; & \widehat{\psi_n}(j\nu;s) & (5)\end{aligned}$$

s^* is the complex conjugate of s. The inverse Fourier transform of $\widehat{\psi_n}(j\nu; s)$ is

$$\psi_n(t;s) = \sqrt{\frac{(2\sigma)^n}{\Gamma(n)}} t^{\frac{n-1}{2}} e^{-s^* t} \tag{6}$$

Define a wavelet transform of an analytic signal $\hat{f}(j\nu)$ as

$$(\mathcal{W}_{\widehat{\psi_n}} \hat{f})(s) = \left\langle \hat{f}(j\nu), \widehat{\psi_n}(j\nu; s) \right\rangle \tag{7}$$

This wavelet transform resolves analytic signals onto the s-plane. The wavelet transform resolves the identity, meaning it has an inverse. Daubechies [1] calls it an isometry from the Hardy space $\mathbf{H}^2$(of analytic signals) onto the Bergman space of analytic functions on the upper half plane. By the Parseval identity, one can expect a time domain version of this wavelet transform. Now we are ready to define the continuous numerical Laplace transform.

Definition 1
Let $f(t) \in L^2(\mathbf{R}^+)$ be a causal signal. Then the n_{th} order numerical Laplace transform of $f(t)$ written as $(\mathcal{L}_{\psi_n} f)(s)$ is

$$\begin{aligned} (\mathcal{L}_{\psi_n} f)(s) &= \langle f(t), \psi_n(t;s) \rangle & (8) \\ &= \sqrt{\frac{(2\sigma)^n}{\Gamma(n)}} \int_0^\infty dt f(t) t^{\frac{n-1}{2}} e^{-st} & (9) \end{aligned}$$

where

$$\sigma > 0, \omega \in \mathbf{R}, n > 1$$

□

This transform resolves causal signals onto the s-plane. We emphasize that this is not quite the usual Laplace transform, because the kernel is essentially $t^{(n-1)/2} e^{-st}$ for $n > 1$, not e^{-st}. Rewrite the kernel $\psi_n(t;s)$ as

$$\begin{aligned} \psi_n(t;s) &= \sigma^{-\frac{1}{2}} \psi_n(\sigma t) e^{j\omega t} & (10) \\ &= \gamma^{1/2}(t; n, 2\sigma) e^{j\omega t} & (11) \end{aligned}$$

One sees that the kernel $\psi_n(t;s)$ is a complex exponential with "delay" n and "damping" σ.

The inverse continuous numerical Laplace transform is [5]

$$f = \frac{(n-1)}{4\pi} \int_0^\infty \frac{d\sigma}{\sigma^2} \int_{-\infty}^\infty d\omega \, \langle f(t), \psi_n(t;s) \rangle \, \psi_n(t;s) \tag{12}$$

3 Time-frequency Resolution of the Continuous Numerical Transform

The continuous numerical Laplace transform has good time-frequency resolution. One way to see this is to view this transform as a variable windowed Fourier transform [5] with $\{\sigma^{-\frac{1}{2}} \psi_n(\sigma t)\}$ or $\{\gamma^{1/2}(t; n, 2\sigma)\}$ as the family of window functions. Each window function $\sigma^{-\frac{1}{2}} \psi_n(\sigma t)$ has centers t^* and ω^* and radii Δ_t and Δ_ω in the time domain and the frequency domain respectively:

$$t^* = \frac{n}{2\sigma} \tag{13}$$
$$\omega^* = 0 \tag{14}$$
$$\Delta_t = \frac{\sqrt{n}}{2\sigma} \tag{15}$$
$$\Delta_\omega = \frac{2\sigma}{2\sqrt{n-2}} \tag{16}$$

Therefore the resolution cell defined by the window function $\sigma^{-\frac{1}{2}} \psi_n(\sigma t)$ centered at $(n/2\sigma, \omega)$ is

$$\left[\frac{n}{2\sigma} - \frac{\sqrt{n}}{2\sigma}, \frac{n}{2\sigma} + \frac{\sqrt{n}}{2\sigma}\right] \times \left[\omega - \frac{2\sigma}{2\sqrt{n-2}}, \omega + \frac{2\sigma}{2\sqrt{n-2}}\right]$$

The size of a resolution cell is thus

$$\begin{aligned} (2\Delta_t)(2\Delta_\omega) &= 2\frac{\sqrt{n}}{\sqrt{n-2}}, \quad n > 2 \\ &\longrightarrow 2, \quad \text{as} \quad n \longrightarrow \infty \end{aligned}$$

Note that this size varies as a function of n. When n increases it approaches 2 which is the lower bound of the Heisenberg Uncertainty Principle which can only be achieved by Gabor transform [2].

The resolution cells are not rigid. They narrow in time resolution when damping is large and widen when damping is small. This is exactly what one would hope to have in the frequency-damping modal analysis. Furthermore, the ratio of the "center-time" to the "time width" is

$$\frac{n/2\sigma}{2\sqrt{n}/2\sigma} = \frac{\sqrt{n}}{2},$$

which is indepent of the damping of the window function. This is analogous to the *constant-Q* property in wavelet analysis. A plot of the tiling is shown in Figure 1.

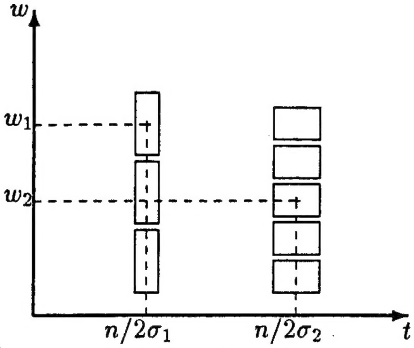

Figure 1: The tiling of the NLT

4 Discretized Numerical Laplace Transform and Frame Bound Estimation

The continuous numerical Laplace transform can be discretized to a lattice in the s-plane, namely $s_{mk} = \sigma_m + j\omega_k$. A suitable choice is $\sigma_m = \sigma_0^m$ and $\omega_k = \sigma_0^m k\omega_0$ for some $\sigma_0 > 0$ and $\omega_0 > 0$. The discretized numerical Laplace transform is defined as follows.

Definition 2
Let $f(t) \in L^2(\mathbf{R}^+)$ be a causal signal and $\psi_n(t; s_{mk}) = \sqrt{\frac{(2\sigma_0^m)^n}{\Gamma(n)}} t^{\frac{n-1}{2}} e^{-s^*_{mk} t}$, $s_{mk} = \sigma_0^m + j\sigma_0^m k\omega_0, \sigma_0 > 0, \omega_0 > 0$. Then the n_{th} order discretized numerical Laplace transform of $f(t)$ is

$$\begin{aligned} & (\mathcal{L}_{\psi_n} f)(m,k) \\ = & \langle f(t), \psi_n(t; s_{m,k}) \rangle \quad (17) \\ = & \sqrt{\frac{(2\sigma_0^m)^n}{\Gamma(n)}} \int_0^\infty dt f(t) t^{\frac{n-1}{2}} e^{-s_{mk} t} \quad (18) \end{aligned}$$

where

$$m, k \in \mathbf{Z}$$

□

The discretized numerical Laplace transform defined in the Definition 2 is invertible only when the $\{\phi_{mk} = \psi_n(t; s_{mk})\}$ constitute a frame, i. e. when there exist frame bounds $0 < A \leq B$ such that

$$A\|f\|^2 \leq \sum_{mk} |\langle f, \phi_{mk} \rangle|^2 \leq B\|f\|^2 \quad (19)$$

for all f in the underlying Hilbert space [1]. Every function f in the underlying Hilbert space can then be decomposed and reconstructed as

$$f = \sum_{mk} \langle f, \phi_{mk} \rangle \widetilde{\phi_{mk}} = \sum_{mk} \left\langle f, \widetilde{\phi_{mk}} \right\rangle \phi_{mk}. \quad (20)$$

ω_0	A	B	B/A
.25	36.256	36.262	1.000
.50	18.127	18.132	1.000
.75	12.060	12.112	1.004
1.00	8.947	9.182	1.026
1.25	6.982	7.522	1.077
1.50	5.594	6.492	1.160

(a)

ω_0	A	B	B/A
.25	13.019	14.138	1.083
.50	6.546	7.092	1.083
.75	4.364	4.728	1.083
1.00	3.223	3.596	1.116
1.25	2.001	3.454	1.726
1.50	0.325	4.221	12.986

(b)

Table 1: Frame bounds for (a) numerical Laplace frames based on ψ_3 and (b) wavelet frames based on Maxican hat function $M(t) = 2/\sqrt{3}\pi^{-1/4}(1 - t^2)e^{-t^2/2}$. The dilation prameter $\sigma_0 = 2$ in both cases.

where $\widetilde{\phi_{mk}}$ is called the dual frame of ϕ_{mk}. When $A = B$ we say the frame is tight and the dual frame is just the frame itself scaled by the inversion of the frame bound. In this case

$$f = \frac{1}{A} \sum_{mk} \langle f, \phi_{mk} \rangle \phi_{mk} \quad (21)$$

When a frame is nearly tight, i. e. A is close to B, equation (21) serves as a good approximation.

Since $\{\widehat{\psi_n}(j\nu; s)\}$ define a continuous wavelet transform , the existence of numerical Laplace frames is guaranteed at least for sufficiently small ω_0 [1]. More than this, it turns out that numerical Laplace frames are nearly tight for a fairly wide range of choices of σ_0, ω_0, and order n. Table 1 gives the frame bound estimations for numerical Laplace frames of order 3 and the wavelet frames based on the Maxican hat function [1]. (the frame bound estimations associated with the Maxican hat function is borrowed from [1]). Frame bound estimates for numerical Laplace frames for other choices of order n and σ_0 are also given in Table 2, 3, 4, 5.

ω_0	A	B	B/A
.25	18.105	18.154	1.003
.50	9.050	9.079	1.003
.75	5.985	6.102	1.020
1.00	4.339	4.726	1.089
1.25	3.253	3.999	1.229
1.50	2.473	3.570	1.443

Table 2: Frame bounds for numerical Laplace frames based on ψ_5 with $\sigma_0 = 2$

ω_0	A	B	B/A
.25	12.004	12.169	1.014
.50	5.996	6.090	1.016
.75	3.917	4.140	1.057
1.00	2.745	3.299	1.202
1.25	1.956	2.878	1.471
1.50	1.394	2.634	1.889

Table 3: Frame bounds for numerical Laplace frames based on ψ_7 with $\sigma_0 = 2$

ω_0	A	B	B/A
.25	22.678	23.067	1.018
.50	11.338	11.539	1.018
.75	7.540	7.712	1.023
1.00	5.578	5.586	1.051
1.25	4.321	4.829	1.118
1.50	3.425	4.200	1.226

Table 4: Frame bounds for numerical Laplace frames based on ψ_3 with $\sigma_0 = 3$

ω_0	A	B	B/A
.25	17.402	18.855	1.083
.50	8.700	9.428	1.084
.75	5.783	6.302	1.090
1.00	4.268	4.796	1.124
1.25	3.294	3.957	1.201
1.50	2.952	3.451	1.332

Table 5: Frame bounds for numerical Laplace frames based on ψ_3 with $\sigma_0 = 4$

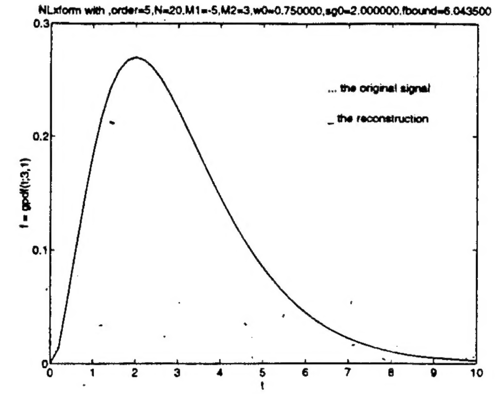

Figure 2: Numerical Laplace frame reconstruction of $\gamma(t;3,1)$ with $n = 5, N = 20, M1 = -5, M2 = 3, \omega_0 = 0.75, \sigma_0 = 2$, bound $= (A+B)/2$

5 Numerical Experiments

Since the numerical Laplace frames are nearly tight, the tight frame decomposition and reconstruction formula (21) can be used as a good approximation with the tight frame bound approximated by $(A+B)/2$ of the numerical Laplace frames. For simplicity we demonstrate two cases: (1) $f(t) = \gamma(t;3,1)$ for a numerical Lapace frame of order 5 and (2) $f(t) = sin(t)$ for a numerical Lapace frame of order 7. For each case we present the experiment results with two plots. One is the plot of the original signal and its numerical Laplace reconstruction. The other is the plot of numerical laplace coefficients in gray scales over the frequency-damping plane. while the former demonstrates the good approximation of tight frame expansions, the later shows how the energy of the signal is distributed over the time-frequency plane in numerical Laplace frame representations.

6 Frequency-Damping Resolution of the Unit Disc

Define damping coefficient ρ and warped frequency θ such that

$$\rho = e^{-\sigma}, \quad \rho \in (0,1) \tag{22}$$

$$\theta = 2\tan^{-1}\omega, \quad \theta \in (-\pi, \pi) \tag{23}$$

Define $z = \rho e^{j\theta}$. Then, via this mapping, the signal is resolved into the frequency-damping unit disc in the z-plane. We may call the corresponding transform in z variable the numerical Z transform except that one

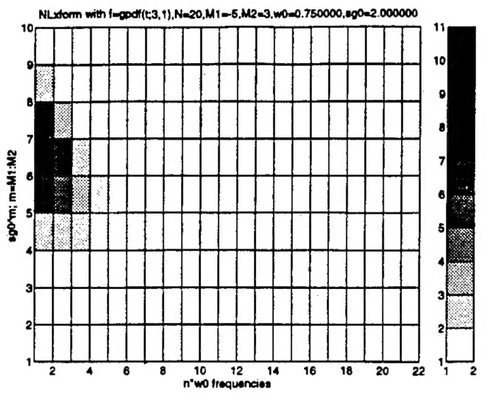

Figure 3: The numerical Laplace coefficients in gray scales of $\gamma(t;3,1)$ with $n=5$,$N=20$,$M1=-5$,$M2=3$,$\omega_0=0.75$,$\sigma_0=2$,bound $=(A+B)/2$

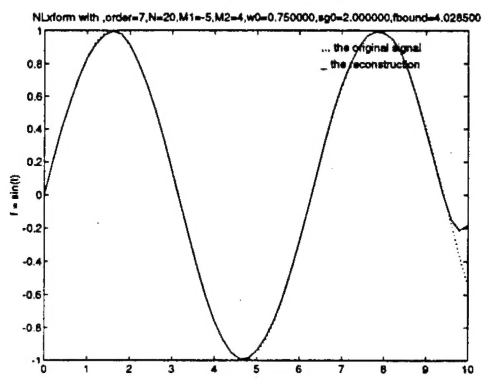

Figure 4: Numerical Laplace frame reconstruction of $\gamma(t;3,1)$ with $n=7$,$N=20$,$M1=-5$,$M2=4$,$\omega_0=0.75$,$\sigma_0=2$,bound $=(A+B)/2$

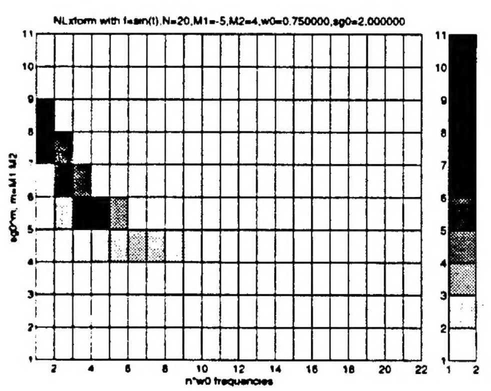

Figure 5: The numerical Laplace coefficients in gray scales of $\gamma(t;3,1)$ with $n=7$,$N=20$,$M1=-5$,$M2=4$,$\omega_0=0.75$,$\sigma_0=2$,bound $=(A+B)/2$

should be reminded that this transform takes continuous signals as input.

Figure 6 shows the lattice after the mapping. The locations of the resolving points are just where we would expect them to be: finely spaced in angular frequency when damping is small and widely spaced when damping is large. This should serve a good model for frequency-damping modal analysis.

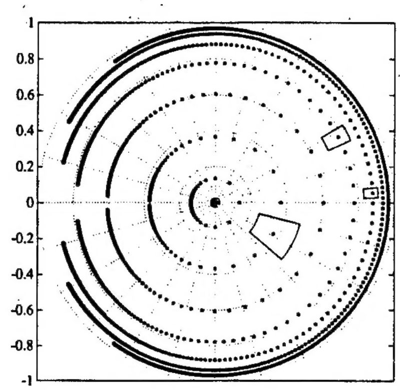

Figure 6: Resolving points of the numerical Z transform.

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